Chapter IV : Small deformations

I Universal families

An analytic family $\pi : \mathcal{X} \to B$ of compact complex manifold is said to be:

- complete at $0 \in B$ if, for every such family $p : \mathcal{Y} \to T$, for every $t \in T$, for every biholomophism $f_0 : p^{-1}(t) \to X_0 = \pi^{-1}(0)$, there exists an open neighborhood $\mathcal{U}$ of t in T and a diagram of holomorphic maps

$$\begin{array}{ccc} p^{-1}(\mathcal{U}) & \xrightarrow{f} & \mathcal{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{U} & \xrightarrow{g} & B \end{array}$$

such that $g(t) = 0$

and $\forall u \in \mathcal{U}, \quad f_{|p^{-1}(u)} \xrightarrow[\text{bihol.}]{\sim} X_{g(u)} = \pi^{-1}(g(u))$.

The family $p_{|\mathcal{U}}$ is analytically isomorphic to the pullback familly of π by g.

- versal at 0 if it is complete at 0 and, furthermore, for every data as before, $dg_{|t} : T_tT \to T_0X$ is uniquely defined.
- universal at 0 if it is complete at 0 and furthermore for every data as before, the germ of g at t is uniquely defined.
- a Kuranishi family of X_0 if it is complete at each point of B an versal at 0.

II Links with the Kodaira–Spencer map.

Theorem

Let $\pi : \mathcal{X} \to B$ be a family.

1) If $KS_0 : T_0B \to \check{H}^1(X_0, \Theta_0)$ is surjective then π is complete at 0.

2) π is versal at 0 <u>iff</u> it is complete at every point of a neighborhood of 0 and KS_0 is isomorphic.

The aim of the rest of the course is to give ideas for

Theorem (Kodaira – Nirenberg – Spencer)

Let X be a compact complex manifold with $\check{H}^2(X,\Theta) = 0$.

Then there exists a Kuranishi family for X parametrized by a ball in $\check{H}^1(X,\Theta)$ i.e. there exists a family $\pi : \mathcal{X} \to B \subset \check{H}^1(X,\Theta)$

with
- $X_0 = \pi^{-1}(0) \underset{\text{bihol.}}{\simeq} X$
- π complete at every point of B
- $KS_0 : T_0B \to \check{H}^1(X,\Theta)$ is the identity.

<u>In general</u>, a family defined over analytic spec$\left(\mathbb{C}[t]/(t^2)\right)$

$\{\theta\} \in H^{0,1}_{\bar\partial}(X,TX)$ by ~~$\Theta \in \check{H}^1(X,\Theta)$~~ extends to analytic spec$\left(\mathbb{C}[t]/(t^3)\right)$,

$\theta \in \mathcal{A}^{0,1}(X,TX)$ <u>iff</u> $\{[\theta,\theta]\} = 0 \in H^{0,2}_{\bar\partial}(X,TX)$.

This gives equations on $H^{0,1}(X,TX)$ for a Kuranishi family.

III Elliptic theory for the Laplacian

- Let X be a compact complex manifold. Choose (by partition of unity) a hermitian metric g on it (that is, a smooth section of $T^{1,0} \otimes T^{0,1}$)

$$g \overset{U^\alpha}{\simeq} \sum g^\alpha_{ij}\, dz^\alpha_i \otimes d\overline{z}^\alpha_j$$

with $$g_{ji} = \overline{g_{ij}}$$

and $$g_{ij}(z)\xi_i\xi_j > 0 \quad \forall z \in U^\alpha \quad \forall (\xi_i) \in \mathbb{C}^n \setminus \{0\}.$$

- Extend this metric to measure forms:
  - * pointwise: If $\left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\right)$ is orthonormal at z_0, then $(d\overline{z})_I = (d\overline{z}_{i_1} \wedge \dots \wedge d\overline{z}_{i_q})_I$ is declared to be an orthonormal basis of $T^{0,q}$ at z_0.
  - * globally: If $\varphi \in \Gamma(X, A^{0,q})$,
    $|\varphi(z_0)|_g$ defined pointwise

$$\|\varphi\|^2_g := \int_X |\varphi(z_0)|^2_g \,|\det g^\alpha_{ij}|^2 dx^\alpha_1 \dots dx^\alpha_n dy^\alpha_1 \dots dy^\alpha_n$$

$\hookrightarrow$ compact.

This gives a hermitian scalar product on $\Gamma(X, A^{0,q})$.

- This enables us to define adjoint operators.

$$\overline{\partial}^*: \Gamma(U, A^{0,q}) \longrightarrow \Gamma(U, A^{0,q-1})$$

defined by: for $\varphi \in \Gamma(U, A^{0,q})$, $\psi \in \Gamma(U, A^{0,q-1})$

$$\langle\langle \overline{\partial}^*\varphi, \psi \rangle\rangle_g \overset{\text{def.}}{=} \langle\langle \varphi, \overline{\partial}\psi \rangle\rangle_g$$

$\overline{\partial}^*$ is also a differential operator of ~~degree~~ order 1.
Let $\overline{\Delta} := (\overline{\partial} + \overline{\partial}^*)^2 = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ is a differential operator of order 2, self adjoint for $\langle\langle\ ,\ \rangle\rangle_g$, elliptic

(its principal symbol is $T^*X \longrightarrow \mathbb{C}$

$$\xi \longmapsto -|\xi|^2$$

never vanishing on $T^*X - \{0\}$), Its kernel is finite dimensional, denoted by $\mathbb{H}$, called space of harmonic forms.

Thanks to ellipticity, we get a decomposition

$$\Gamma(X, \mathcal{A}^{0,q}) = \mathbb{H}^{0,q} \oplus \bar{\Delta}\,\Gamma(X, \mathcal{A}^{0,q})$$
$$= \mathbb{H}^{0,q} \oplus \bar{\partial}\,\Gamma(X, \mathcal{A}^{0,q-1}) \oplus \bar{\partial}^*\,\Gamma(X, \mathcal{A}^{0,q+1})$$

$\operatorname{Ker} \bar{\partial} = \mathbb{H}^{0,q} \oplus \bar{\partial}\,\Gamma(X, \mathcal{A}^{0,q-1})$ because

$$\langle\langle \bar{\partial}\bar{\partial}^* \psi, \psi \rangle\rangle_g = \|\bar{\partial}^* \psi\|_g^2$$

$H^{0,q}_{\bar{\partial}}(X, \mathbb{C}) \simeq \mathbb{H}^{0,q}$ (Hodge ~~decomposition~~ representation theorem)

(In every Dolbeault cohomology class, there is a unique g - harmonic representative).

Now, take $\varphi \in \Gamma(X, \mathcal{A}^{0,q})$

$$\varphi = \mathbb{H}(\varphi) + \bar{\Delta}\varphi = \mathbb{H}(\varphi) + \bar{\Delta}\big(\mathbb{H}(\psi) + \overset{\bar{\Delta}\eta}{\cancel{\bar{\partial}\bar{\partial}^*\psi + \bar{\partial}^*\bar{\partial}\psi}}\big)$$
$$= \mathbb{H}(\varphi) + \bar{\Delta}\big(\cancel{\bar{\partial}\bar{\partial}^*\psi}\ \underbrace{\bar{\Delta}\eta}_{\in \mathbb{H}^\perp}\big)$$

There exists an operator $G : \Gamma(X, \mathcal{A}^{0,q}) \longrightarrow (\mathbb{H}^{0,q})^\perp \subset \Gamma(X, \mathcal{A}^{0,q})$ such that $\forall \varphi \in \Gamma(X, \mathcal{A}^{0,q})$

$$\varphi = \mathbb{H}(\varphi) + \bar{\Delta}(G\varphi)$$

G is called the Green operator.

We have some estimates:

Define $\|\cdot\|_{g,k}$ a norm on $\Gamma(X, \mathcal{A}^{0,q})$ that takes into accounts all partial derivatives up to order k.

$\exists c>0, \forall \varphi \in \Gamma(X, \mathcal{A}^{0,q})$

$$\|\bar\Delta\varphi\|_k \leq c\|\varphi\|_{k+2}$$ (the Laplacian is of order 2)

$$\|\varphi\|_{k+2} \leq c(\|\bar\Delta\varphi\|_k + \|\varphi\|_0)$$ Laplacian is "almost invertible"

$$\|G\varphi\|_k \leq c\|\varphi\|_{k-2}$$ G almost an inverse for Δ "smoothing' property".

IV Sketch of proof for the Kodaira-Nirenberg-Spencer thm

Let X be a compact complex manifold with $\check{H}^2(X,\Theta)=0$.
Let g be an hermitian metric on X, also used as a metric on TX.
Fix $(\eta_1,\dots,\eta_m)$ a basis for $H^{0,1}(X,TX)$.
(In the previous section, we could have worked with forms $\Gamma(X,\mathcal{A}^{0,q}(E))$ with values in a hermitian differentiable vector bundle).

TX

Set, for $t\in\mathbb{C}^m$, $\varphi_1(t) = \sum_{i=1}^m t_i\eta_i \in H^{0,1}(X,TX)\subset\Gamma(X,\mathcal{A}^{0,1}(TX))$
We would like to construct $\varphi(t)\in\Gamma(X,\mathcal{A}^{0,1}(TX))$ such that $\bar\partial_{X_0}\varphi(t) = \frac{1}{2}[\varphi(t),\varphi(t)]$ (**)

Thanks to the following theorem, this will give a new complex structure X_t on $(X)_{dif}$.

Theorem (Newlander - Nirenberg)
Let U be an open subset of $\mathbb{C}^n$,
$\varphi = \varphi_{ij}\, d\bar z_i \frac{\partial}{\partial z_j} \in \Gamma(U,\mathcal{A}^{0,1}(TU))$

Set
$$\bar\partial_i^\varphi := \frac{\partial}{\partial \bar z_i} - \varphi_{ij}\frac{\partial}{\partial z_j}$$
$$\partial_i^\varphi := \frac{\partial}{\partial z_i} - \overline{\varphi_{ij}}\frac{\partial}{\partial \bar z_j}$$
vector fields on $\mathbb{C}^n$

If $(\bar\partial_i^\varphi, \partial_i^\varphi \;,\; 1 \le i \le n,\ 1 \le j \le n)$ are $\mathbb{C}$-linearly independent and if $\bar\partial^\varphi \circ \bar\partial^\varphi = 0$, where $\bar\partial^\varphi = \bar\partial_i^\varphi \, d\bar z_i$, then there exists smooth functions $\zeta_1, \dots, \zeta_n$ on U such that the Jacobian $\dfrac{\partial(\zeta_1 \dots \zeta_n, \bar\zeta_1 \dots \bar\zeta_n)}{\partial(z_1 \dots z_n, \bar z_1 \dots \bar z_n)}$ never vanishes on U and $\bar\partial^\varphi \zeta_j = 0$.

Remark

$\bar\partial^\varphi \circ \bar\partial^\varphi = 0 \iff \bar\partial\varphi = \frac{1}{2}[\varphi, \varphi]$. ///

We will in fact consider the equation (instead of (**))

$$\varphi(t) = \varphi_1(t) + \frac{1}{2} \bar\partial^* G\, [\varphi(t), \varphi(t)] \qquad (***)$$

Expanding $\varphi(t)$ as $\sum_{|\nu|=1}^{+\infty} \varphi_\nu t^\nu$ $\quad \nu = (\nu_1, \dots, \nu_m)$, $|\nu| = \sum_{i=1}^{m} \nu_i$

$$\varphi_2(t) = \cancel{\varphi_1(t)} + \frac{1}{2} \bar\partial^* G\, [\varphi_1(t), \varphi_1(t)]$$

$$\varphi_\nu(t) = \sum_{\substack{p+q=\nu \\ p,q \ge 1}} \frac{1}{2} \bar\partial^* G\, [\varphi_p(t), \varphi_q(t)]$$

We have to check the convergence of $\sum \varphi_\nu t^\nu$ for this recursive choice of φ_ν.

$$\begin{aligned}
\|\varphi_\nu\|_k &= \Big\| \frac{1}{2} \sum \bar\partial^* G\, [\varphi_p, \varphi_q] \Big\|_k \\
&\le c \Big\| \sum G\, [\varphi_p, \varphi_q] \Big\|_{k+1} \\
&\le c \, \| [\varphi_p, \varphi_q] \|_{k-1} \\
&\le c \, \|\varphi_p\|_k \, \|\varphi_q\|_k
\end{aligned}$$

This is enough to prove the $\mathcal{C}^k$ convergence for $|t| \ll 1$.
We get $\varphi(t)$ a smooth $(0,1)$-form on X with values in TX satisfying $(***)$.

Lemma

If $\quad \mathbb{H}[\varphi(t), \varphi(t)] = 0 \in \Gamma(X, \mathcal{A}^{0,2}(T_*X))$,
then $\quad \bar\partial \varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)]$

Proof

Set $\quad \psi = \bar\partial\varphi(t) - \frac{1}{2}[\varphi(t), \varphi(t)]$

$$
\begin{aligned}
2\psi &= \bar\partial\left(2\varphi_1 + \tfrac{1}{2}\bar\partial^* G[\varphi,\varphi]\right) - [\varphi,\varphi] \\
&= \bar\partial\bar\partial^* G[\varphi,\varphi] - [\varphi,\varphi]
\end{aligned}
$$

(because φ_1 is $\bar\Delta$-harmonic, hence $\bar\partial$-closed.)

$$
\begin{aligned}
&= \bar\partial\bar\partial^* G[\varphi,\varphi] - \underbrace{\mathbb{H}[\varphi,\varphi]}_{=0 \text{ assumption}} - \underbrace{\bar\Delta}_{\bar\partial\bar\partial^* + \bar\partial^*\bar\partial} G[\varphi,\varphi] \\
&= -\bar\partial^*\bar\partial G[\varphi,\varphi] \\
&= -\bar\partial^* G(2[\bar\partial\varphi,\varphi]) \quad \text{because } G\bar\partial = \bar\partial G.
\end{aligned}
$$

$$
\begin{aligned}
\psi &= -\bar\partial G^*[\bar\partial\varphi,\varphi] = -\bar\partial^* G^*[\psi,\varphi] + \tfrac{1}{2}\bar\partial G^* \underbrace{[[\varphi,\varphi],\varphi]}_{=0} \\
&= -\bar\partial^* G[\psi,\varphi]
\end{aligned}
$$

Jacobi formula

$\|\psi\|_k \leq c\|\psi\|_k \|\varphi\|_k$

If $t \ll 1$, then $\quad c\|\varphi\|_k < 1$, hence $\psi = 0$.

Remark By Hodge representation theorem

$$\mathbb{H}^{0,2}(X, TX) \simeq \check{H}^2(X, \Theta) \quad \text{assumed to be } 0.$$

(Hence the lemma's hypothesis is fullfilled).
This gives a complex structure X_t on $(X)_{\text{diff}} \underset{\substack{\uparrow \\ \text{notation}}}{=} M$.

To construct the family $\mathcal{X} \longrightarrow B$, we will construct a complex structure on $M \times B$.

$$\Phi(m,t) := \varphi(t)(m) \qquad \varphi(t) \in \Gamma(X, A^{0,1}(TX))$$

$$\Phi \in \Gamma(X \times B, A^{0,1}(TX \times B))$$

$\varphi(t)$ is analytic in t by convergence. $\bar{\partial}_B \varphi(t) = 0$.

Hence $\bar{\partial}_{X \times B} \Phi = \bar{\partial}_X \Phi = \bar{\partial}_X \varphi(t) = \frac{1}{2}[\varphi, \varphi] = \frac{1}{2}[\Phi, \Phi]$

By Newlander-Nirenberg theorem, we get a complex structure $\mathcal{X}$ on $X \times B$; whose local holomorphic functions are solutions of $\bar{\partial}_{X \times B} f - \Phi f = 0$.

In particular, $t_1, \dots, t_m$ are holomorphic functions, and give a hol. map $\pi : \mathcal{X} \longrightarrow B$, proper, submersion. $\pi^{-1}(t) = X_t$ by the choice of Φ.

$$\frac{\partial X_t}{\partial t_j} = \frac{\partial \varphi(t)}{\partial t_j} = \frac{\partial \varphi_1}{\partial t} = \eta_j$$

KS; $\frac{\partial}{\partial t_j} \longrightarrow \eta_j$ is surjective : π is complete at 0

is injective : π is versal at 0

Using $H^{0,2}(X, TX) = 0$, we can show using semi-continuity of $t \longmapsto \dim H^1(X_t, TX_t)$ that the family is complete in a neighborhood of 0.

It is a Kuranishi family of X.